Lecture 7

More on Constitute Relations, Uniform Plane Wave

7.1 More on Constitutive Relations

As have been seen, Maxwell's equations are not solvable until the constitutive relations are included. Here, we will look into depth more into various kinds of constitutive relations. Now that we have learned phasor technique, we can have a more general constitutive relationship compared to what we have seen earlier.

7.1.1 Isotropic Frequency Dispersive Media

First let us look at the simple linear constitutive relation previously discussed for dielectric media where [30], [31][p. 82], [43]

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{7.1.1}$$

We have a simple model where

$$\mathbf{P} = \varepsilon_0 \chi_0 \mathbf{E} \tag{7.1.2}$$

where χ_0 is the electric susceptibility. When used in the generalized Ampere's law, $\mathbf{P}$, the polarization density, plays an important role for the flow of the displacement current through space. The polarization density is due to the presence of polar atoms or molecules that become little dipoles in the presence of an electric field. For instance, water, which is H_2O, is a polar molecule that becomes a small dipole when an electric field is applied.

We can think of displacement current flow as capacitive coupling yielding polarization current flow through space. Namely, for a source-free medium,

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \tag{7.1.3}$$

Figure 7.1: As a series of dipoles line up end to end, one can see a current flowing through the line of dipoles as they oscillate back and forth in their polarity. This is similar to how displacement current flows through a capacitor.

For example, for a sinusoidal oscillating field, the dipoles will flip back and forth giving rise to flow of displacement current just as how time-harmonic electric current can flow through a capacitor as shown in Figure 7.1.

The linear relationship above can be generalized to that of a linear time-invariant system, or that at any given $\mathbf{r}$ [34][p. 212], [43][p. 330].

$$\mathbf{P}(\mathbf{r}, t) = \varepsilon_0 \chi_e(\mathbf{r}, t) \circledast \mathbf{E}(\mathbf{r}, t) \tag{7.1.4}$$

where $\circledast$ here implies a convolution. In the frequency domain or the Fourier space, the above linear relationship becomes

$$\mathbf{P}(\mathbf{r}, \omega) = \varepsilon_0 \chi_0(\mathbf{r}, \omega)\mathbf{E}(\mathbf{r}, \omega), \tag{7.1.5}$$

or

$$\mathbf{D}(\mathbf{r}, \omega) = \varepsilon_0(1 + \chi_0(\mathbf{r}, \omega))\mathbf{E}(\mathbf{r}, \omega) = \varepsilon(\mathbf{r}, \omega)\mathbf{E}(\mathbf{r}, \omega) \tag{7.1.6}$$

where $\varepsilon(\mathbf{r}, \omega) = \varepsilon_0(1 + \chi_0(\mathbf{r}, \omega))$ at any point $\mathbf{r}$ in space. There is a rich variety of ways at which $\chi_0(\omega)$ can manifest itself. Such a permittivity $\varepsilon(\mathbf{r}, \omega)$ is often called the effective permittivity. Such media where the effective permittivity is a function of frequency is termed dispersive media, or frequency dispersive media.

7.1.2 Anisotropic Media

For anisotropic media [31][p. 83]

$$\begin{aligned} \mathbf{D} &= \varepsilon_0 \mathbf{E} + \varepsilon_0 \overline{\boldsymbol{\chi}}_0(\omega) \cdot \mathbf{E} \\ &= \varepsilon_0(\overline{\mathbf{I}} + \overline{\boldsymbol{\chi}}_0(\omega)) \cdot \mathbf{E} = \overline{\varepsilon}(\omega) \cdot \mathbf{E} \end{aligned} \tag{7.1.7}$$

In the above, $\overline{\boldsymbol{\varepsilon}}$ is a 3×3 matrix also known as a tensor in electromagnetics. The above implies that $\mathbf{D}$ and $\mathbf{E}$ do not necessary point in the same direction, the meaning of anisotropy. (A tensor is often associated with a physical notion like the relation between two physical fields, whereas a matrix is not.)

Previously, we have assume that χ_0 to be frequency independent. This is not usually the case as all materials have χ_0's that are frequency dependent. This will become clear later. Also, since $\overline{\varepsilon}(\omega)$ is frequency dependent, we should view it as a transfer function where the input is $\mathbf{E}$, and the output $\mathbf{D}$. This implies that in the time-domain, the above relation becomes a time-convolution relation as in (7.1.4).

Similarly for conductive media,

$$\mathbf{J} = \sigma \mathbf{E}, \tag{7.1.8}$$

This can be used in Maxwell's equations in the frequency domain to yield the definition of complex permittivity. Using the above in Ampere's law in the frequency domain, we have

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\varepsilon\mathbf{E}(\mathbf{r}) + \sigma\mathbf{E}(\mathbf{r}) = j\omega\underset{\sim}{\varepsilon}(\omega)\mathbf{E}(\mathbf{r}) \tag{7.1.9}$$

where the complex permittivity $\underset{\sim}{\varepsilon}(\omega) = \varepsilon - j\sigma/\omega$. This is the first instance of the use of complex permittivity. Notice that Ampere's law in the frequency domain with complex permittivity in (7.1.9) is no more complicated than Ampere's law for nonconductive media. The algebra for complex numbers is no more difficult than the algebra for real numbers. This is one of the strengths of phasor technique.

For anisotropic conductive media, one has

$$\mathbf{J} = \overline{\boldsymbol{\sigma}}(\omega) \cdot \mathbf{E}, \tag{7.1.10}$$

Here, again, due to the tensorial nature of the conductivity $\overline{\boldsymbol{\sigma}}$, the electric current $\mathbf{J}$ and electric field $\mathbf{E}$ do not necessary point in the same direction.

The above assumes a local or point-wise relationship between the input and the output of a linear system. This need not be so. In fact, the most general linear relationship between $\mathbf{P}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ is

$$\mathbf{P}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \overline{\boldsymbol{\chi}}(\mathbf{r} - \mathbf{r}', t - t') \cdot \mathbf{E}(\mathbf{r}', t') d\mathbf{r}' dt' \tag{7.1.11}$$

In the Fourier transform space, the above becomes

$$\mathbf{P}(\mathbf{k}, \omega) = \overline{\boldsymbol{\chi}}(\mathbf{k}, \omega) \cdot \mathbf{E}(\mathbf{k}, \omega) \tag{7.1.12}$$

where

$$\overline{\boldsymbol{\chi}}(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} \overline{\boldsymbol{\chi}}(\mathbf{r}, t) \exp(j\mathbf{k} \cdot \mathbf{r} - j\omega t) d\mathbf{r} dt \tag{7.1.13}$$

(The $d\mathbf{r}$ integral above is actually a three-fold integral with $d\mathbf{r} = dxdydz$.) Such a medium is termed spatially dispersive as well as frequency dispersive [34][p. 6], [50]. In general[1]

$$\overline{\boldsymbol{\varepsilon}}(\mathbf{k}, \omega) = 1 + \overline{\boldsymbol{\chi}}(\mathbf{k}, \omega) \tag{7.1.14}$$

where

$$\mathbf{D}(\mathbf{k}, \omega) = \overline{\boldsymbol{\varepsilon}}(\mathbf{k}, \omega) \cdot \mathbf{E}(\mathbf{k}, \omega) \tag{7.1.15}$$

The above can be extended to magnetic field and magnetic flux yielding

$$\mathbf{B}(\mathbf{k}, \omega) = \overline{\boldsymbol{\mu}}(\mathbf{k}, \omega) \cdot \mathbf{H}(\mathbf{k}, \omega) \tag{7.1.16}$$

for a general spatial and frequency dispersive magnetic material. In optics, most materials are non-magnetic, and hence, $\mu = \mu_0$, whereas it is quite easy to make anisotropic magnetic materials in radio and microwave frequencies, such as ferrites.

[1]In the following, one should replace the 1 with an identity operator, but it is generally implied.

7.1.3 Bi-anisotropic Media

In the previous section, the electric flux **D** depends on the electric field **E** and the magnetic flux **B**, on the magnetic field **H**. The concept of constitutive relation can be extended to where **D** and **B** depend on both **E** and **H**. In general, one can write

$$\mathbf{D} = \overline{\boldsymbol{\varepsilon}}(\omega) \cdot \mathbf{E} + \overline{\boldsymbol{\xi}}(\omega) \cdot \mathbf{H} \tag{7.1.17}$$

$$\mathbf{B} = \overline{\boldsymbol{\zeta}}(\omega) \cdot \mathbf{E} + \overline{\boldsymbol{\mu}}(\omega) \cdot \mathbf{H} \tag{7.1.18}$$

A medium where the electric flux or the magnetic flux is dependent on both **E** and **H** is known as a bi-anisotropic medium [31][p. 81].

7.1.4 Inhomogeneous Media

If any of the $\overline{\boldsymbol{\varepsilon}}$, $\overline{\boldsymbol{\xi}}$, $\overline{\boldsymbol{\zeta}}$, or $\overline{\boldsymbol{\mu}}$ is a function of position **r**, the medium is known as an inhomogeneous medium or a heterogeneous medium. There are usually no simple solutions to problems associated with such media [34].

7.1.5 Uniaxial and Biaxial Media

Anisotropic optical materials are often encountered in optics. Examples of them are the biaxial and uniaxial media, and discussions of them are often found in optics books [51–53]. They are optical materials where the permittivity tensor can be written as

$$\overline{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \tag{7.1.19}$$

When $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3$, the medium is known as a biaxial medium. But when $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$, then the medium is a uniaxial medium.

In the biaxial medium, all three components of the electric field feel different permittivity constants. But in the uniaxial medium, the electric field in the xy plane feels the same permittivity constant, but the electric field in the z direction feels a different permittivity constant. As shall be shown later, different light polarization will propagate with different behavior through such a medium.

7.1.6 Nonlinear Media

In the previous cases, we have assumed that $\overline{\boldsymbol{\chi}}_0$ is independent of the field **E**. The relationships between **P** and **E** can be written more generally as

$$\mathbf{P} = \varepsilon_0 \overline{\boldsymbol{\chi}}_0(\mathbf{E}) \cdot \mathbf{E} \tag{7.1.20}$$

where the relationship can appear in many different forms. For nonlinear media, the relationship can be nonlinear as indicated in the above. It can be easily shown that the principle of linear superposition does not hold for the above equation, a root test of linearity. Nonlinear permittivity effect is important in optics. Here, the wavelength is short, and a small change in the permittivity or refractive index can give rise to cumulative phase delay as the wave propagates through a nonlinear optical medium [54–56]. Kerr optical nonlinearity, discovered in 1875, was one of the earliest nonlinear phenomena observed [31, 51, 54].

For magnetic materials, nonlinearity can occur in the effective permeability of the medium. In other words,

$$\mathbf{B} = \overline{\boldsymbol{\mu}}(\mathbf{H}) \cdot \mathbf{H} \tag{7.1.21}$$

This nonlinearity is important even at low frequencies, and in electric machinery designs [57, 58], and magnetic resonance imaging systems [59]. The large permeability in magnetic materials is usually due to the formation of magnetic domains which can only happen at low frequencies.

7.2 Wave Phenomenon in the Frequency Domain

We have seen the emergence of wave phenomenon in the time domain. Given the simplicity of the frequency domain method, it will be interesting to ask how this phenomenon presents itself for time-harmonic field or in the frequency domain. In the frequency domain, the source-free Maxwell's equations are [31][p. 429], [60][p. 107]

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu_0 \mathbf{H}(\mathbf{r}) \tag{7.2.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\varepsilon_0 \mathbf{E}(\mathbf{r}) \tag{7.2.2}$$

Taking the curl of (7.2.1) and then substituting (7.2.2) into its right-hand side, one obtains

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu_0 \nabla \times \mathbf{H}(\mathbf{r}) = \omega^2 \mu_0 \varepsilon_0 \mathbf{E}(\mathbf{r}) \tag{7.2.3}$$

Again, using the identity that

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \cdot \nabla \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \tag{7.2.4}$$

and that $\nabla \cdot \mathbf{E} = 0$ in a source-free medium, (7.2.3) becomes

$$(\nabla^2 + \omega^2 \mu_0 \varepsilon_0)\mathbf{E}(\mathbf{r}) = 0 \tag{7.2.5}$$

This is known as the Helmholtz wave equation or just the Helmholtz equation.[2]

For simplicity of seeing the wave phenomenon, we let $\mathbf{E} = \hat{x}E_x(z)$, a field pointing in the x direction, but varies only in the z direction. Evidently, $\nabla \cdot \mathbf{E}(\mathbf{r}) = \partial E_x(z)/\partial x = 0$. Then (7.2.5) simplifies to

$$\left(\frac{d^2}{dz^2} + k_0^2\right) E_x(z) = 0 \tag{7.2.6}$$

where $k_0^2 = \omega^2 \mu_0 \varepsilon_0 = \omega^2 / c_0^2$ where c_0 is the velocity of light. The general solution to (7.2.6) is of the form

$$E_x(z) = E_{0+} e^{-jk_0 z} + E_{0-} e^{jk_0 z} \tag{7.2.7}$$

[2]For a comprehensive review of this topic, one may read the lecture notes [39].

One can convert the above back to the time domain using phasor technique, or by using that $E_x(z,t) = \Re e[E_x(z,\omega)e^{j\omega t}]$, yielding

$$E_x(z,t) = |E_{0+}|\cos(\omega t - k_0 z + \alpha_+) + |E_{0-}|\cos(\omega t + k_0 z + \alpha_-) \tag{7.2.8}$$

where we have assumed that

$$E_{0\pm} = |E_{0\pm}|e^{j\alpha_\pm} \tag{7.2.9}$$

The physical picture of the above expressions can be appreciated by rewriting

$$\cos(\omega t \mp k_0 z + \alpha_\pm) = \cos\left[\frac{\omega}{c_0}(c_0 t \mp z) + \alpha_\pm\right] \tag{7.2.10}$$

where we have used the fact that $k_0 = \frac{\omega}{c_0}$. One can see that the first term on the right-hand side of (7.2.8) is a sinusoidal plane wave traveling to the right, while the second term is a sinusoidal plane wave traveling to the left, with velocity c_0. The above plane wave is uniform and a constant in the xy plane and propagating in the z direction. Hence, it is also called a uniform plane wave in 1D.

Moreover, for a fixed t or at $t = 0$, the sinusoidal functions are proportional to $\cos(\mp k_0 z + \alpha_\pm)$. This is a periodic function in z with period $2\pi/k_0$ which is the wavelength λ_0, or that

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega}{c_0} = \frac{2\pi f}{c_0} \tag{7.2.11}$$

One can see that because c_0 is a humongous number, λ_0 can be very large. You can plug in the frequency of your local AM 920 station to see how big λ_0 is.

The above analysis still holds true if ϵ_0 and μ_0 are replaced by ϵ and μ but are real numbers. In this case, the velocity c of the wave is now given by $c = 1/\sqrt{\mu\epsilon}$. This velocity is the velocity of the phase of a time-harmonic signal and hence, is known as phase velocity.

7.3 Uniform Plane Waves in 3D

By repeating the previous derivation for a homogeneous lossless medium, the vector Helmholtz equation for a source-free medium is given by [39]

$$\nabla \times \nabla \times \mathbf{E} - \omega^2 \mu\varepsilon \mathbf{E} = 0 \tag{7.3.1}$$

By the same derivation as before for the free-space case, one has

$$\nabla^2 \mathbf{E} + \omega^2 \mu\varepsilon \mathbf{E} = 0 \tag{7.3.2}$$

if $\nabla \cdot \mathbf{E} = 0$.

The general solution to (7.3.2) is hence

$$\mathbf{E} = \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{7.3.3}$$

where $\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$, $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$ and $\mathbf{E}_0$ is a constant vector. And upon substituting (7.3.3) into (7.3.2), it is seen that

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon \tag{7.3.4}$$

This is called the dispersion relation for a plane wave.

In general, k_x, k_y, and k_z can be arbitrary and even complex numbers as long as this relation is satisfied. To simplify the discussion, we will focus on the case where k_x, k_y, and k_z are all real numbers. When this is the case, the vector function represents a uniform plane wave propagating in the $\mathbf{k}$ direction. As can be seen, when $\mathbf{k}\cdot\mathbf{r}$ = constant, it is represented by all points of $\mathbf{r}$ that represents a flat plane (see Figure 7.2). This flat plane represents the constant phase wave front. By increasing the constant, we obtain different planes for progressively changing phase fronts.[3]

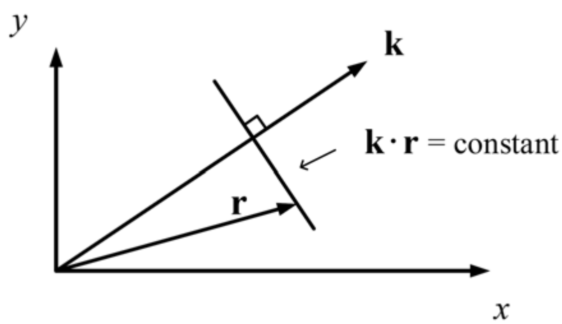

Figure 7.2: A figure showing the geometrical meaning of $\mathbf{k}\cdot\mathbf{r}$ equal to a constant. It is a flat plane that defines the wavefront of a plane wave.

Further, since $\nabla\cdot\mathbf{E} = 0$, we have

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \nabla\cdot\mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z} = \nabla\cdot\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\\ &= (-\hat{x}jk_x - \hat{y}jk_y - \hat{z}jk_z)\cdot\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\\ &= -j(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z)\cdot\mathbf{E} = 0\end{aligned} \tag{7.3.5}$$

or that

$$\mathbf{k}\cdot\mathbf{E}_0 = \mathbf{k}\cdot\mathbf{E} = 0 \tag{7.3.6}$$

Thus, $\mathbf{E}$ is orthogonal to $\mathbf{k}$ for a uniform plane wave.

The above exercise shows that whenever $\mathbf{E}$ is a plane wave, and when the ∇ operator operates on such a vector function, one can do the substitution that $\nabla \to -j\mathbf{k}$. Hence, in a source-free homogenous medium,

$$\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H} \tag{7.3.7}$$

[3]In the $\exp(j\omega t)$ time convention, this phase front is decreasing, whereas in the $\exp(-i\omega t)$ time convention, this phase front is increasing.

the above equation simplifies to

$$-j\mathbf{k}\times\mathbf{E} = -j\omega\mu\mathbf{H} \tag{7.3.8}$$

or that

$$\mathbf{H} = \frac{\mathbf{k}\times\mathbf{E}}{\omega\mu} \tag{7.3.9}$$

Similar to (7.3.3), we can define

$$\mathbf{H} = \mathbf{H}_0 e^{-jk_x x - jk_y y - jk_z z} = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{7.3.10}$$

Then using (7.3.3) in (7.3.9), it is clear that

$$\mathbf{H}_0 = \frac{\mathbf{k}\times\mathbf{E}_0}{\omega\mu} \tag{7.3.11}$$

We can assume that $\mathbf{E}_0$ and $\mathbf{H}_0$ are real vectors. Then $\mathbf{E}_0$, $\mathbf{H}_0$ and $\mathbf{k}$ form a right-handed system, or that $\mathbf{E}_0\times\mathbf{H}_0$ point in the direction of $\mathbf{k}$. (This also implies that $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{k}$ form a right-handed system.) Such a wave, where the electric field and magnetic field are transverse to the direction of propagation, is called a transverse electromagnetic (TEM) wave.

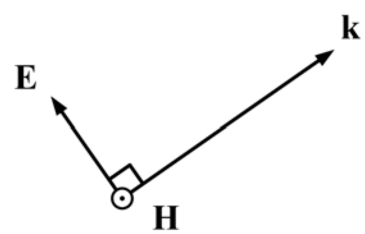

Figure 7.3: The $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{k}$ together form a right-hand coordinate system, obeying the right-hand rule. Namely, $\mathbf{E}\times\mathbf{H}$ points in the direction of $\mathbf{k}$.

Figure 7.3 shows that $\mathbf{k}\cdot\mathbf{E} = 0$, and that $\mathbf{k}\times\mathbf{E}$ points in the direction of $\mathbf{H}$ as shown in (7.3.9). Figure 7.3 also shows, as $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{H}$ are orthogonal to each other.

Since in general, $\mathbf{E}_0$ and $\mathbf{H}_0$ can be complex vectors, because they are phasors, we need to show the more general case. From (7.3.9), one can show, using the "back-of-the-cab" formula, that

$$\mathbf{E}\times\mathbf{H}^* = \mathbf{E}\cdot\mathbf{E}^*\frac{\mathbf{k}}{\omega\mu} = |\mathbf{E}|^2\frac{\mathbf{k}}{\omega\mu} \tag{7.3.12}$$

But $\mathbf{E}\times\mathbf{H}^*$ is the direction of power flow, and it is in fact in the $\mathbf{k}$ direction. This is also required by the Poynting's theorem.

Furthermore, we can show in general that

$$|\mathbf{H}| = \frac{|\mathbf{k}||\mathbf{E}|}{\omega\mu} = \sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}| = \frac{1}{\eta}|\mathbf{E}| \tag{7.3.13}$$

where the quantity

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{7.3.14}$$

is call the intrinsic impedance. For vacuum or free-space, it is about 377 $\Omega \approx 120\pi$ Ω.

In the above, when k_x, k_y, and k_z are not all real, the wave is known as an inhomogeneous wave.[4]

[4]The term inhomogeneous plane wave is used sometimes, but it is a misnomer since there is no more a planar wave front in this case.